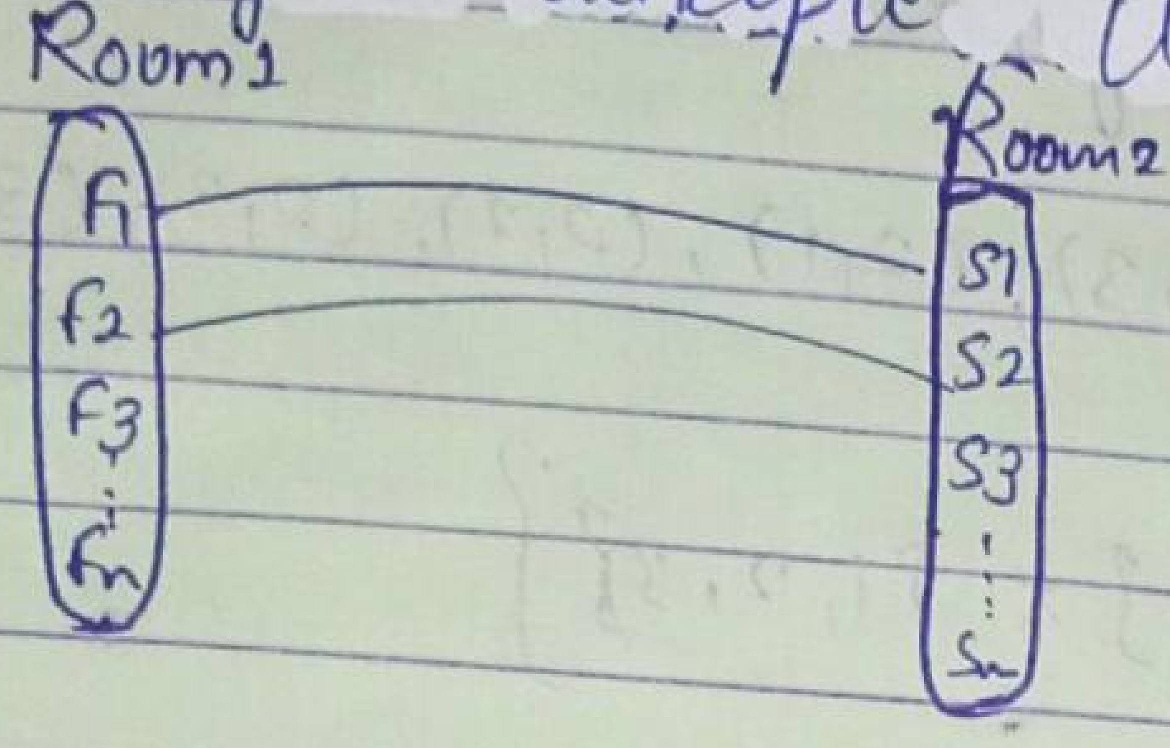
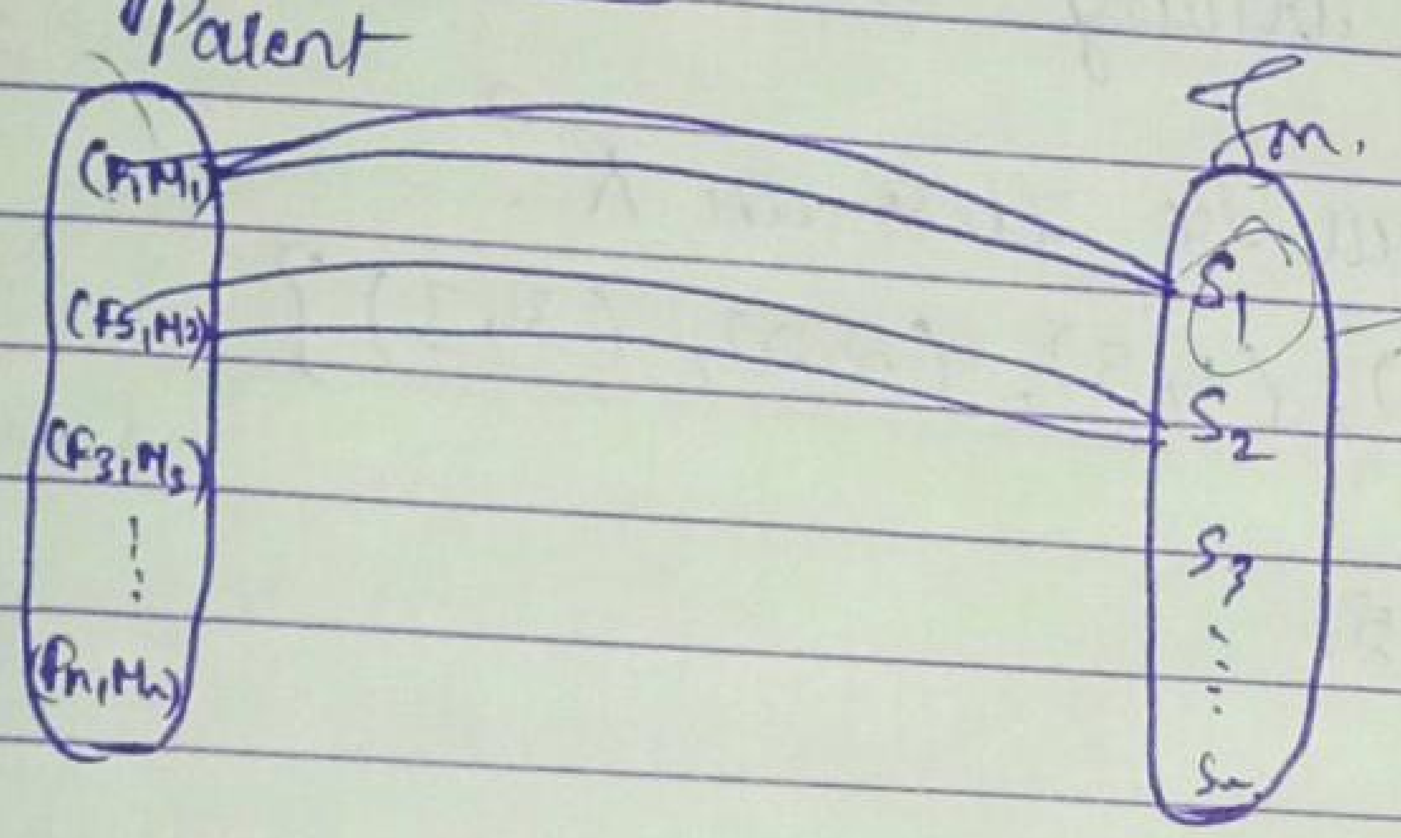
Counting Principle and Relation



Binary relation –



$A = \{a_1, a_2, \dots a_m\}$

$A \times A \rightarrow A$

$\{(a_1, a_2)(a_3, a_4), (a_1, a_4) \dots \}$

Ex:– $A = \{1, 2\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

Domain set – $A \times A$ Codomain set – A

Sep 11

Binary relation

Def: A binary relation from the set A to the set B is a subset of $A \times B$.

Ex:– $A = \{1, 2\}$ $B = \{1, 2\}$

$A \times B = \{(1,1), (1,2), (2,1), (2,2)\}$

Ex :– $A = \{1, 2\}$ $B = \{2, 3\}$

$A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$

Eg:- A binary relation is defined on the set $A = \{1, 2, 3\}$.

Solⁿ:- $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$\{A \times A = \{\{1, 2, 3\} \times \{1, 2, 3\}\}\}$

Ex. $A = \{1, 2, 3\}$ define a binary relation $R = \{(a,b) \mid a \text{ divides } b\}$

How many elements will be there in R?

Solⁿ:- $R = \{(1,2), (1,1), (1,3), (2,2), (3,3)\}$

No. of elements = 5.

Properties

① Reflexivity / Reflexive Relation

→ A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$.

Ex:- $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4)\}$

Is R reflexive?

⇒ No., because (4,4) is missing.

Note:- For a relation to be reflexive, All element present in set A must be present in R in form of (a,a).

3}.

2),(3,3) }

|a
divides b}

∈ R

Symmetric and Antisymmetric

→ A relation R on a set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a,b \in A$. A relation R on a set A such that for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$ is called antisymmetric.

Ex:- $(1,2) \mid (2,1)$

$R = \{(1,2), (2,1)\}$ is symmetric? Yes.

Ex $R = \{(1,2), (2,1), (3,1)\}$ is symmetric? No., because mirror image of $(3,1)$ is not in relation.

Ex. $R = \{(1,2), (2,1), (1,1)\}$ is Symmetric? Yes.

Ex $R = \{(1,1), (2,2), (3,3)\}$ is Symmetric? Yes, every ~~one~~ element is same as their image.

Ex.- $A = \{1,2,3\}$
$R = \{(1,1), (1,2), (2,1)\}$ is symmetric??

→ yes, R is symmetric.

Note:- For a relation set to be symmetric, only elements present inside the set must mirror image.

Antisymmetric

Ex:- $R = \{(1,1), (1,2), (2,1)\}$ is antisymmetric? No., because mirror image of $(1,2)$ is present in R

Ex: $R = \{(1,1), (1,2)\}$ is antisymmetric?

Yes.

Ex: $R = \{(1,2), (1,3), (2,3)\}$ is antisymmetric?

→ Yes., because no ~~two~~ mirror image of any element present inside the relation set.

Ex: $R = \{(1,2), (1,3), (3,1), (2,3)\}$ is antisymmetric?

→ No., because mirror image of (1,3) is present inside the relation.

Note: This set is also not symmetric. (bcoz mirror image of (2,3) is not present in relation R)

Ex: $R = \{(1,1), (2,2), (3,3)\}$ is this symmetric?

→ Yes, R is symmetric, antisymmetric and reflexive also.

Ex: $R = \{(a,b) \mid a \geq b\}$

Is it reflexive?? Is it symmetric??

→ Yes bcoz, equality symbol is there.

→ No, let $a = 2$, $b = 1$ (2,1) (1,2)

$(2,1) \in R$ but $(1,2) \notin R$

$\therefore$ R is not symmetric.

(3)

Q1 $A = \{1, 2, 3\}$

$R = \{(1,1), (2,1), (1,2)\}$

a) It is reflexive ?

→ No.

b) Is it symmetric ?

→ Yes.

c) Is it antisymmetric

→ No.

Q2 $R = \{(1,1), (2,1), (1,2), (2,2)\}$

a) Is is Symmetric ?

→ No.

b) Is it anti-symmetric ?

→ No.

Transitive Relation

Def:- If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$, where R is the binary relation defined in a set A.

Ex:- $R = \{(a,b) \mid a \le b\}$

let $(a,b) \in R$ and $\Rightarrow a \le b$
$(b,c) \in R$ $\Rightarrow b \le c$ $\Big\} \Rightarrow a \le c$

Thus, R is transitive. $(a,c) \in R$

Ex:- $R = \{(a,b) \mid a = b\}$

let $(a,b) \in R \Rightarrow a = b$
$(b,c) \in R \Rightarrow b = c$ $\Big\} \Rightarrow a = c$

Thus R is transitive

Ex. $R = \{(a,b) \mid a = b+1\}$ Is this transitive ??

Sol: let $(a,b) \in R \Rightarrow a = b+1 \Rightarrow b = a-1$

$(b,c) \in R \Rightarrow b = c+1 \Rightarrow a-1 = c+1 \Rightarrow a = c+2$
$= (c+1)+1$

Non transitive.

Matrix Representation of Binary Relation.

Ex

$A = \{1,2,3\}$

$R = \{(1,1), (2,3), (3,1), (3,2)\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & (1,1) & (1,2) & (1,3) \\ 2 & (2,1) & (2,2) & (2,3) \\ 3 & (3,1) & (3,2) & (3,3) \end{array} \qquad \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array}$$

Ex $A = \{1,2,3\}$ $R = \{(a,b) \mid a \le b\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\}$

Matrix representation

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{array}$$

Is Reflexive ?? Yes.
Is Symmetric ?? No
Is anti-symmetric ?? Yes
Is transitive ??

For Reflexive → All diagonal element must be 1.

For Symmetric → upper triangular and lower triangular matrix must be image imge of each other

For anti-symmetric → No mirror image

Q
$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is this reflexive → Yes, diagonal elements are 1

Is this symmetric → Yes, upper Δ ~~matr~~ is mirror image of lower matrix.

Is this Anti-symmetric → No.

14th Sep.

Q
$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Transitive only
(b) Reflexive only
(c) Ref, Sym, and anti-sym.
(d) Sym, but not anti-symmetric.

Ex.

$$M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 \end{array}$$

Is it transitive ??

No. $(1,2) \in R$ but $(1,3) \notin R$
$(2,3) \in R$

Thus, Non-Transitive

Equivalence Relation:

Def:- A binary relation is called an equivalence relation, if it is reflexive, symmetric and transitive.

Ex:- $R = \{(a,b) \mid a=b \text{ or } a=-b\}$

Is R a equivalence relation?

Sol^n R is reflexive
R is symmetric
R is transitive
$\Rightarrow$ R is an equivalence relation

Ex. # Partial Order Relation:

Def:- A binary relation is called partial order relation, if it is reflexive, anti-symmetric and transitive.

Def:- A set 'S' together with a partial order relation R is called partially ordered set or POSET.

~~Ex. A = {1, 2, 3}~~

$R = \{(a,b) \mid a \le b\}$

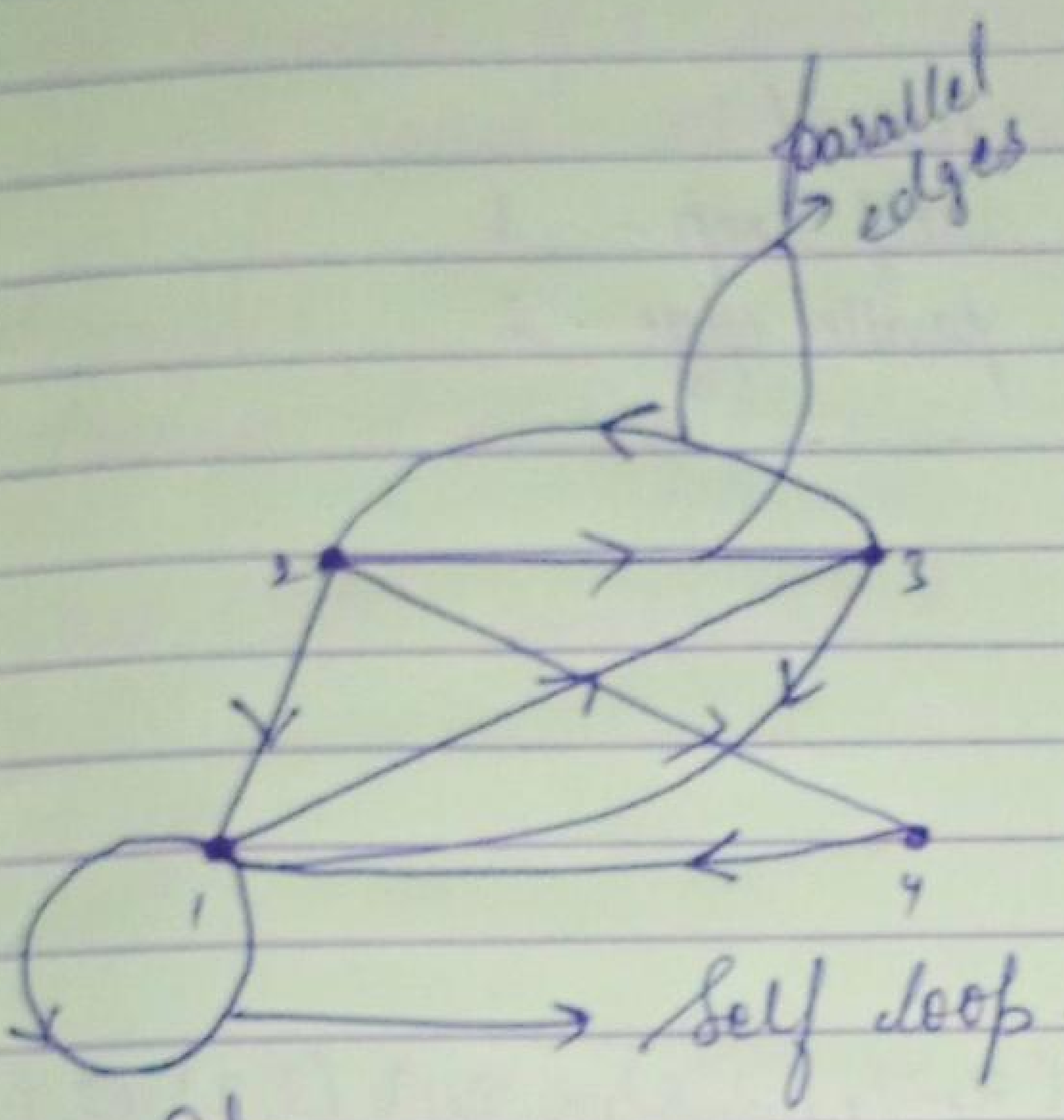
Ex Relation ($\ge$). is it a partial order relation in the set of integers?

$\rightarrow$ Reflexive, ~~symmetric~~ transitive and anti-symmetric
$\Rightarrow$ Relation ($\ge$) is a partial order relation.

Graphical Representation of Binary Relation

Ex. $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

So, (1, 2, 3, 4) are vertices

Joining lines are called edges.

18th Sep.

Q $A = \{1, 2, 3 .... 10\}$

- How many self loop will be there in the graphical representation of the binary relation $R = \{(a,b) \mid a \leq b\}$?

(a) 8 (b) 9 (c) 10 (d) 11 (e) None.

Note:- If a relation is reflexive, then no of self loop must be equal to the number of points in the base set.

Q R = {(1,1), (2,2), (1,3), (2,3), (3,1)}

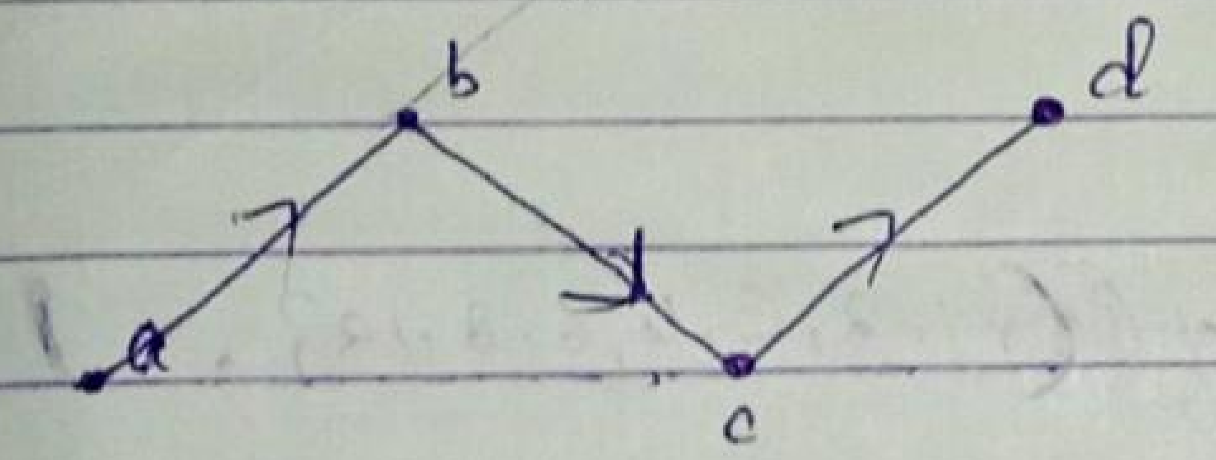
Q How many self loops will be found in the group of the binary representation.

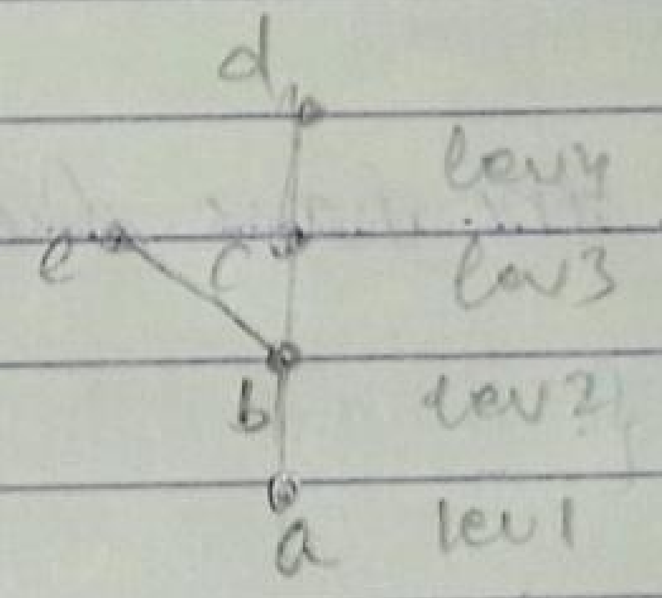
(a) 1 (b) 2 (c) 3 (d) 4 (e) None

Q2 How many parallel edges you have?

a) 1 (b) 2 (c) 3 (d) 4 (e) None

Hasse Diagram — Simplest graphical representation of a poset.





"Direction is always upward".

Hasse diagram is only applicable in POSET".

POSET → Partially ordered set.

1. Reflexive — (1,1), (2,2)
2. Anti-symmetric — (1,3) but (3,1) not be in relation
3. Transitive. — (1,3) (3,4), (1,4)

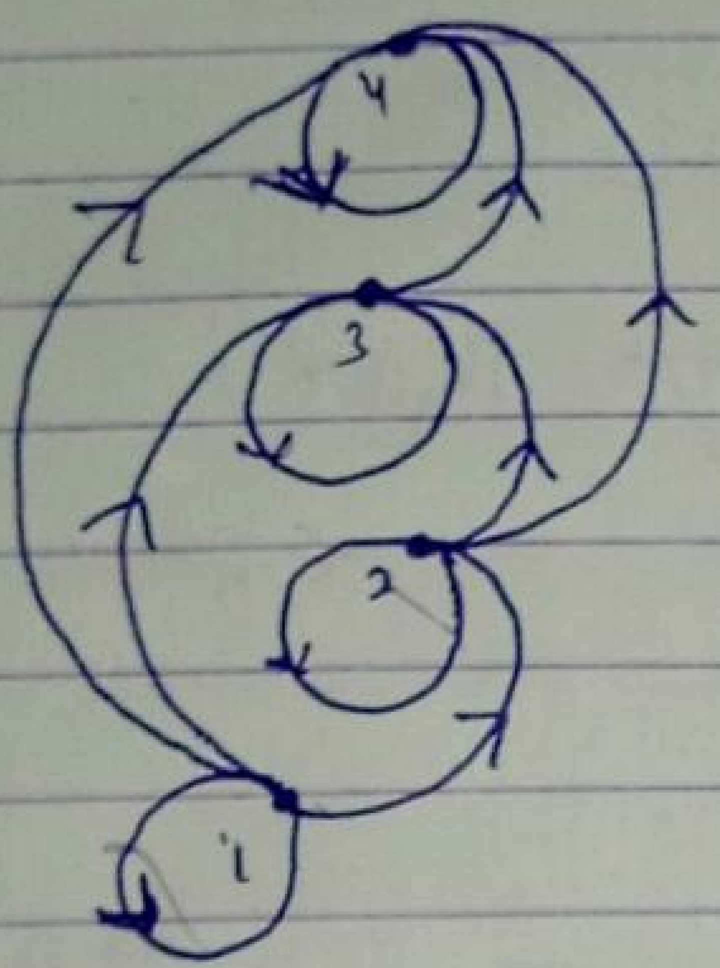
(100)

Smriti Roy

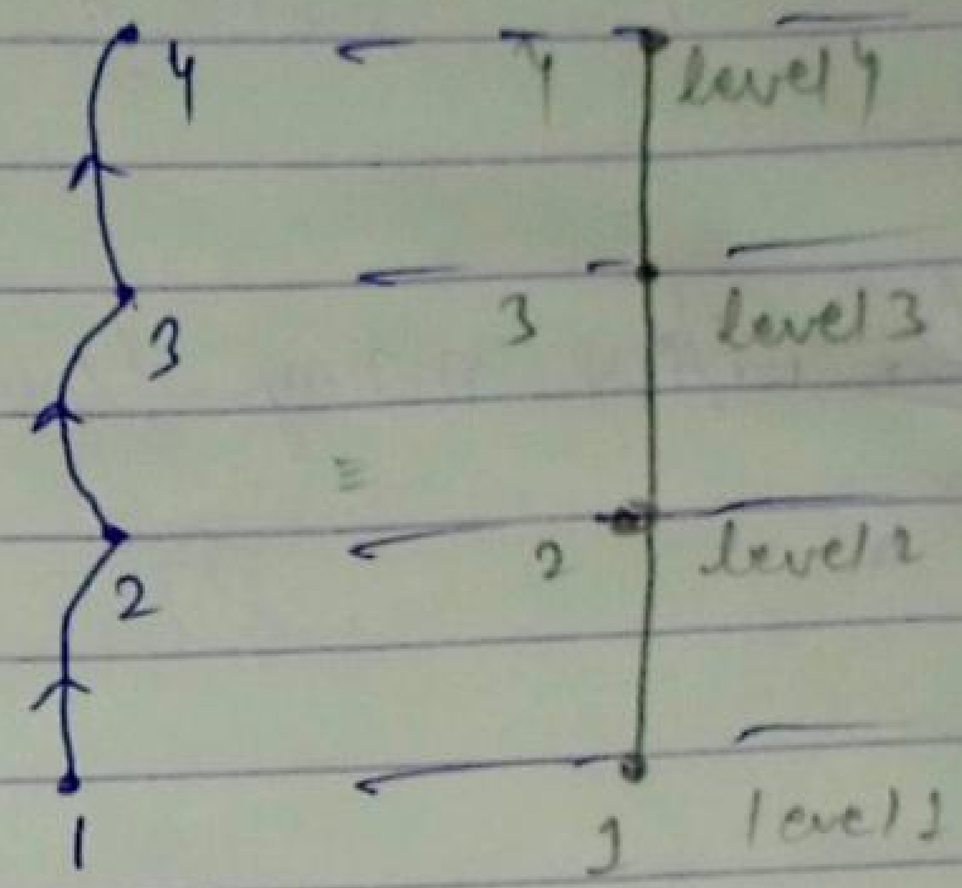
Q $\{ \{1, 2, 3, 4\} , \leq \}$

$R = \{ (1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,3), (2,4)$
$(3,4) \}$

Construct a hasse diagram.



≡

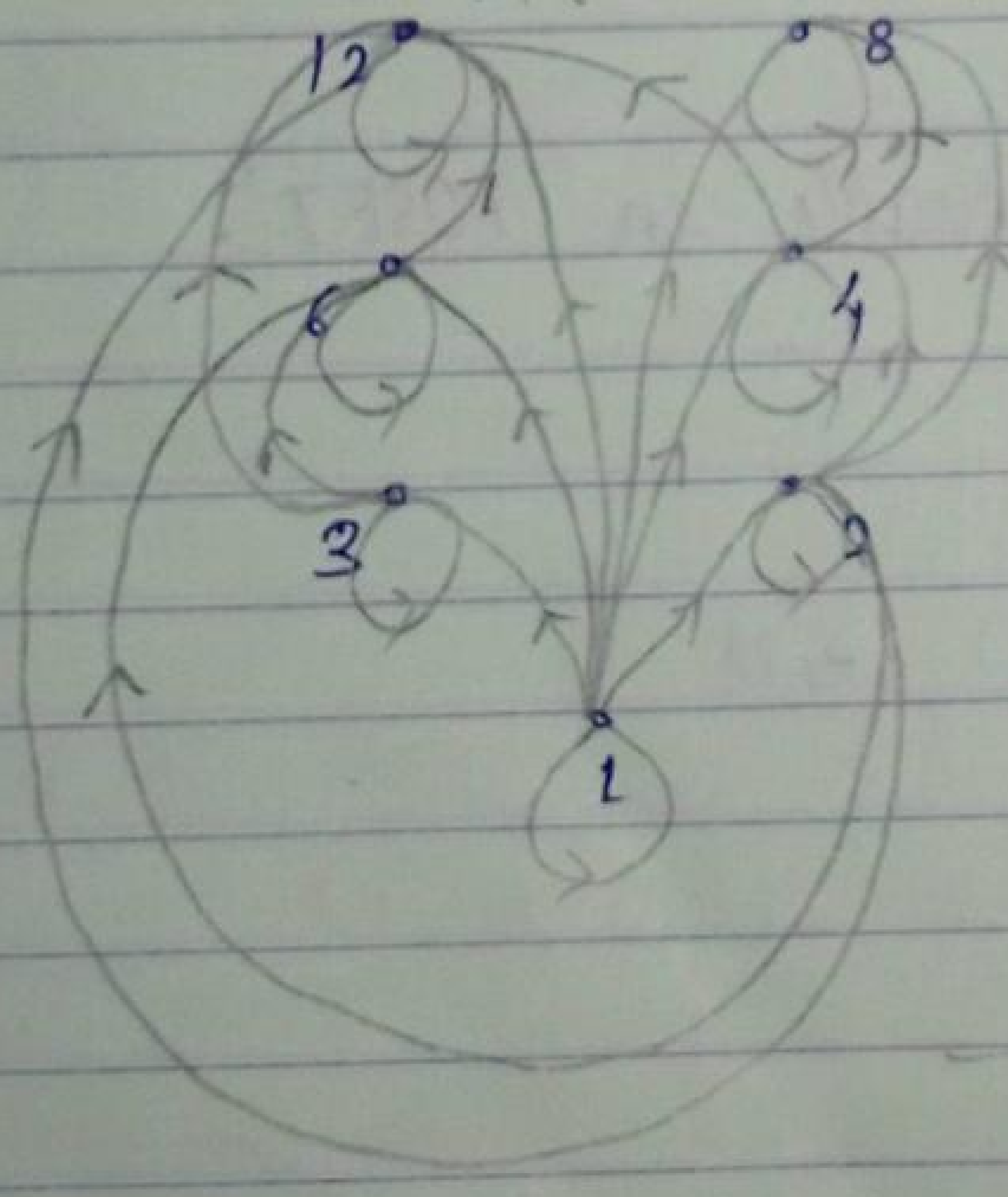


Hasse Diagram

21/09/2021

Q Construct the hasse Diagram of $(\{1, 2, 3, 4, 6, 8, 12\}, \mid)$

Solⁿ:- $R = \{(1,1), (2,2), (3,3), (4,4), (6,6), (8,8), (12,12), (1,2), (1,3), (1,4), (1,6), (1,8),$
$(1,12), (2,4), (2,6), (2,8), (2,12), (3,6), (3,12), (4,8), (4,12), (6,12)\}$



≡

$1 \to 2 \to 4$ (1,4)

$1 \to 6 \to 12$ (1,12)

$1 \to 3 \to 6$ (1,6)

$3 \to 6 \to 12$ (3,12)

$2 \to 4 \to 8$ (2,8)

$2 \to 6 \to 12$ (2,12)

$1 \to 4 \to 8$ (1,8)

(1,1)

(2,2)

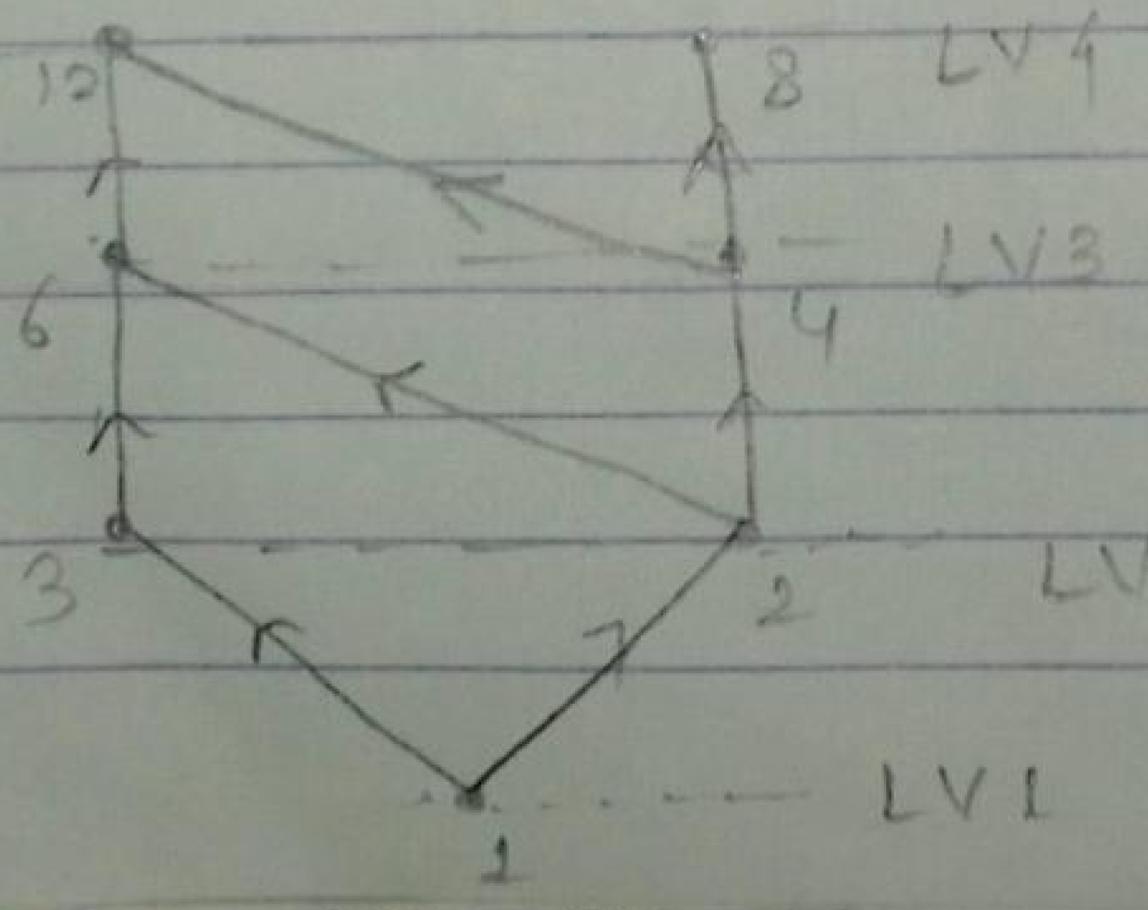
(3,3)

(4,4)

(6,6)

(8,8)

(12,12)



Comparability

The elements a and b of a poset $(S, \leq)$ are called comparable, if either $a \leq b$ or $b \leq a$. When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, a and b are called non-comparable.

Example:
Consider $(Z^+, /)$ $Z^+ \rightarrow$ Set of all +ve integers

are $(3, 9)$ comparable? are 5 and 3 a comparable

$\rightarrow$ yes. No.

Ex $(Z^+, \leq)$

is 1 and 3 comparable?

$\rightarrow$ Yes. $(1 \leq 3)$.

Hesse Diagram

Q R = {(1,1), (2,2), (1,3), (2,3), (3,1)}

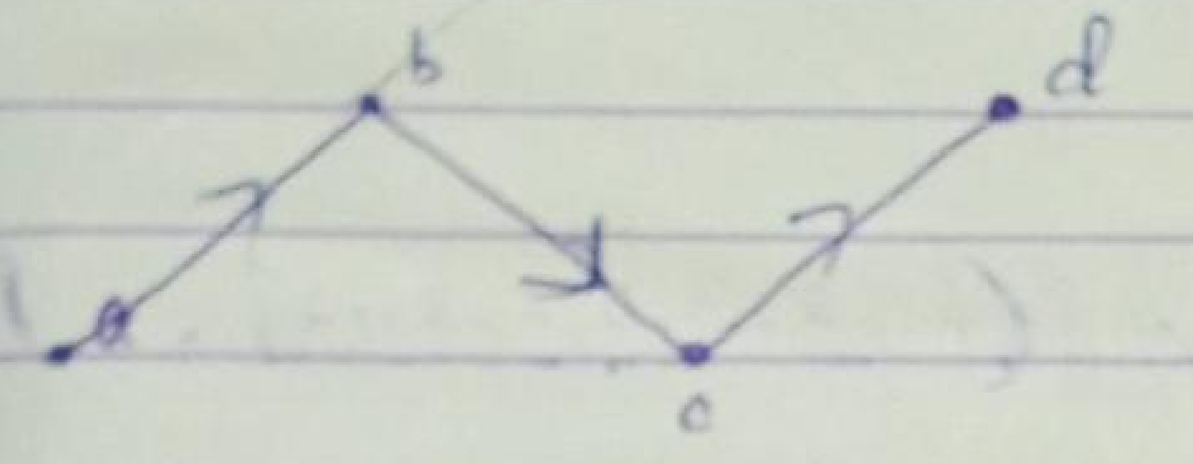
Q How many self loops will be found in the group of the binary representation.

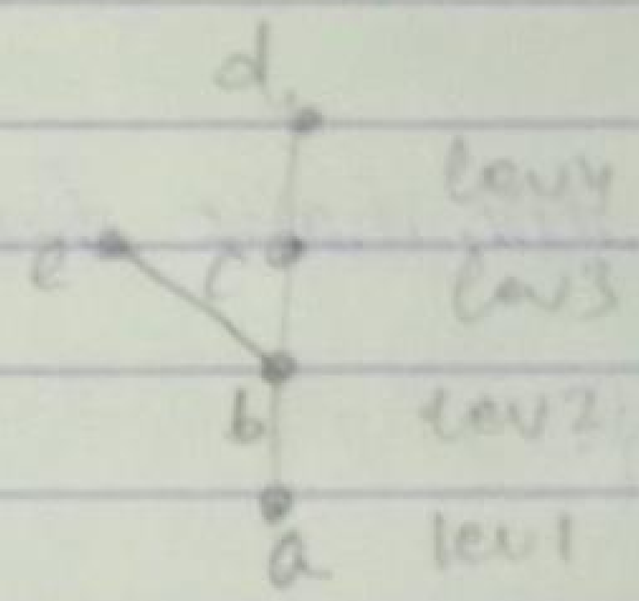
a) 1 b) 2 c) 3 d) 4 e) None

Q2 How many parallel edges you have?

a) 1 b) 2 c) 3 d) 4 e) None

Hasse Diagram — Simplest graphical representation of a poset.





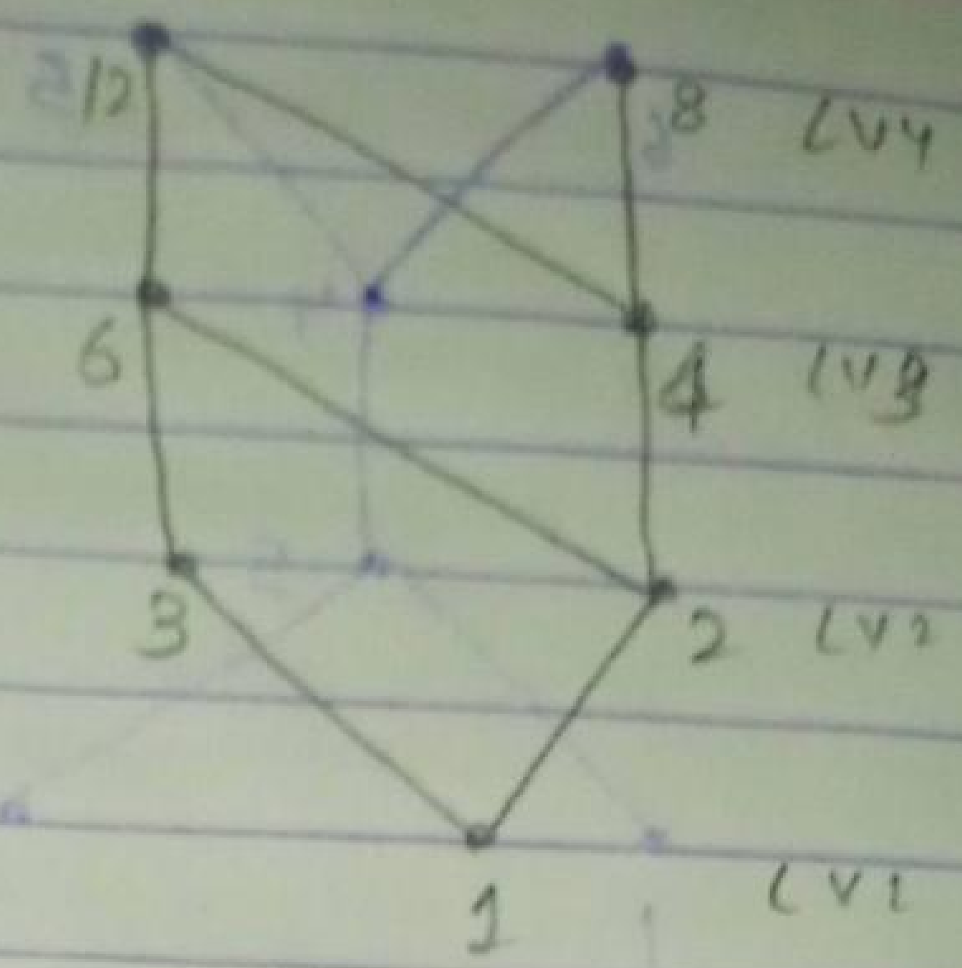
"Direction is always upward".

Hasse diagram is only applicable in POSET".

POSET → Partially ordered set.

1. Reflexive — (1,1), (2,2)
2. Anti-symmetric — (1,3) but (3,1) not be in relation
3. Transitive — (1,3)(3,4), (1,4)

hesse diagram



Ex 2.

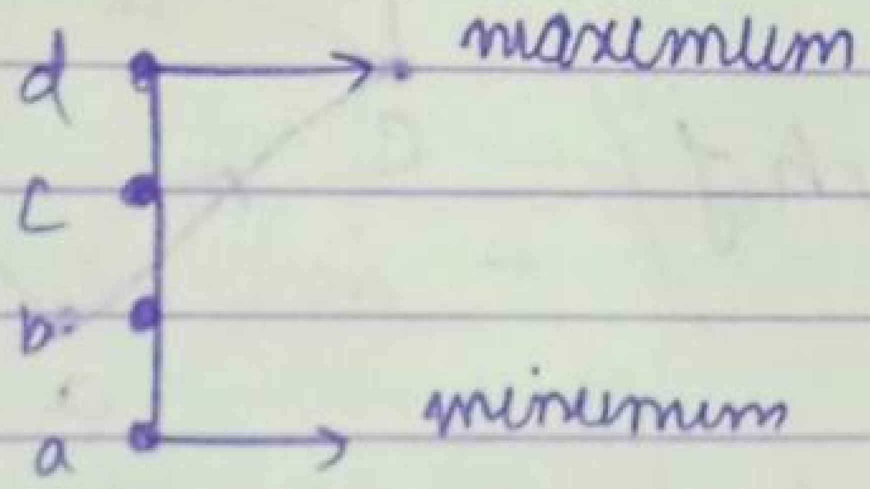
points: ① Delete all self loops
② : Delete the transitive cases.

Properties of Hasse Diagram

① <u>Maximum</u> and <u>Minimum</u> element → <u>Single point</u>

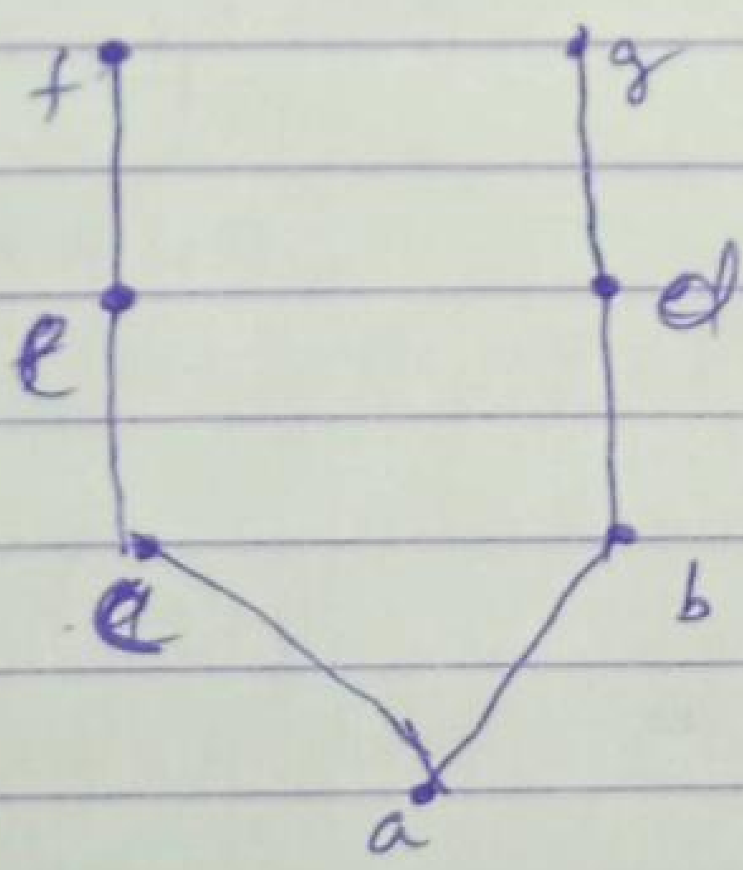
② <u>Maximal</u> and <u>Minimal</u> element → Related to multiple element.

Ex :- 1



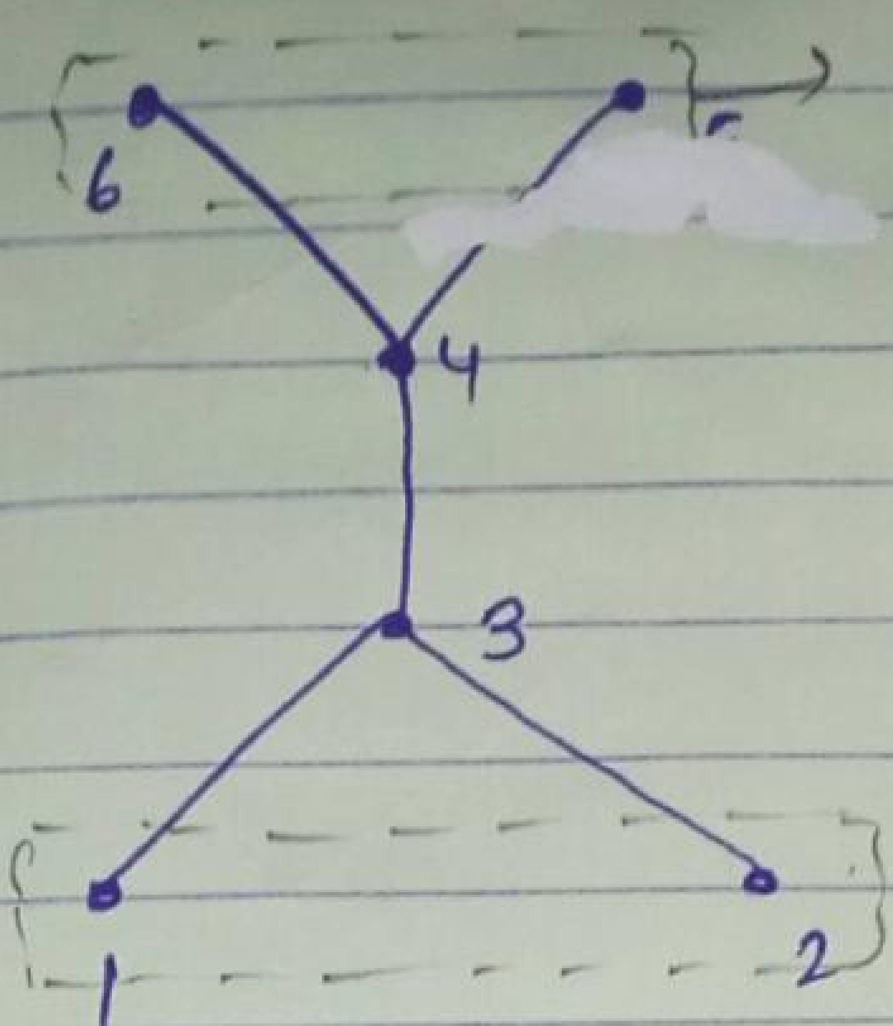
which is max or min Element ?

Ex 2.



Minimum → a
Maximal → f & g
Maximum - None

Ex 3.

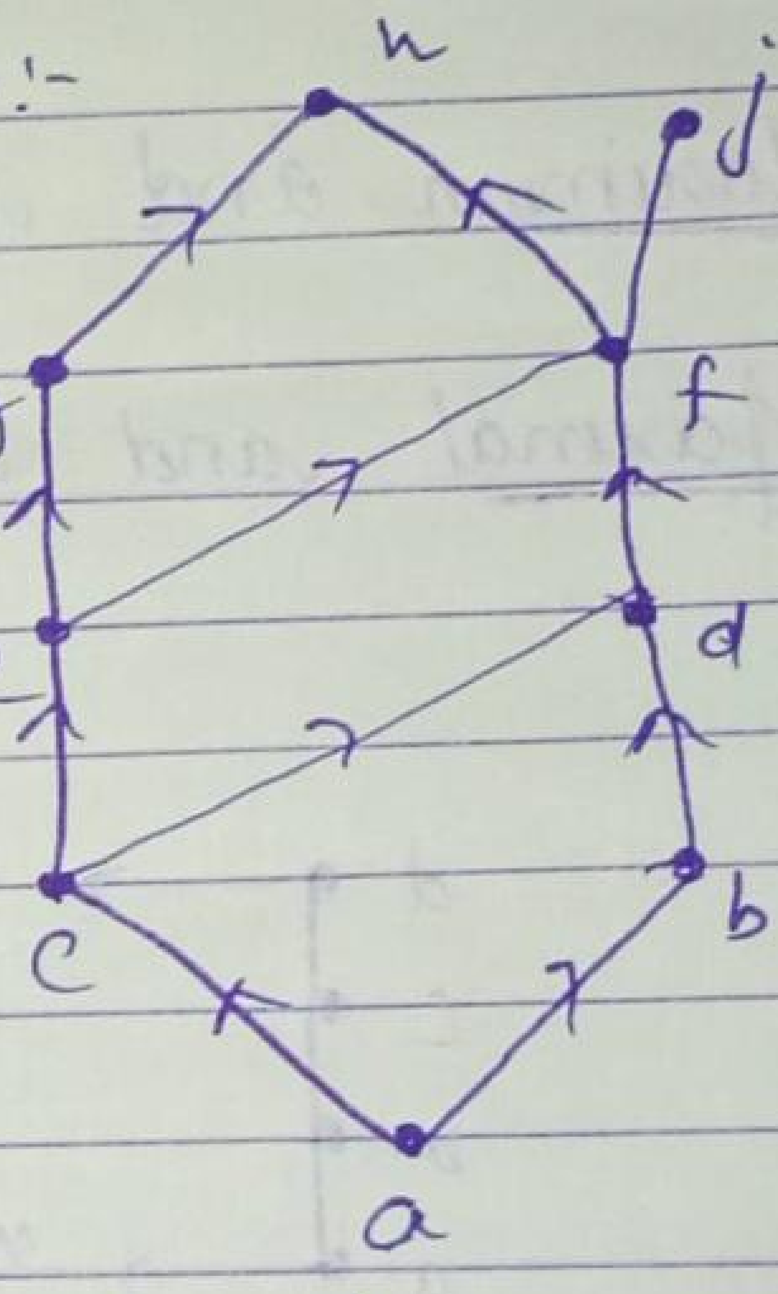


Maximum element → None
Minimum element → None
Maximal element → 5, 6
Minimal element → 1, 2

Point :- Maximum is also called greatest element
Minimum is also called least element

Upper and Lower Bounds.

Ex :-



upper bound of (a, b, c)

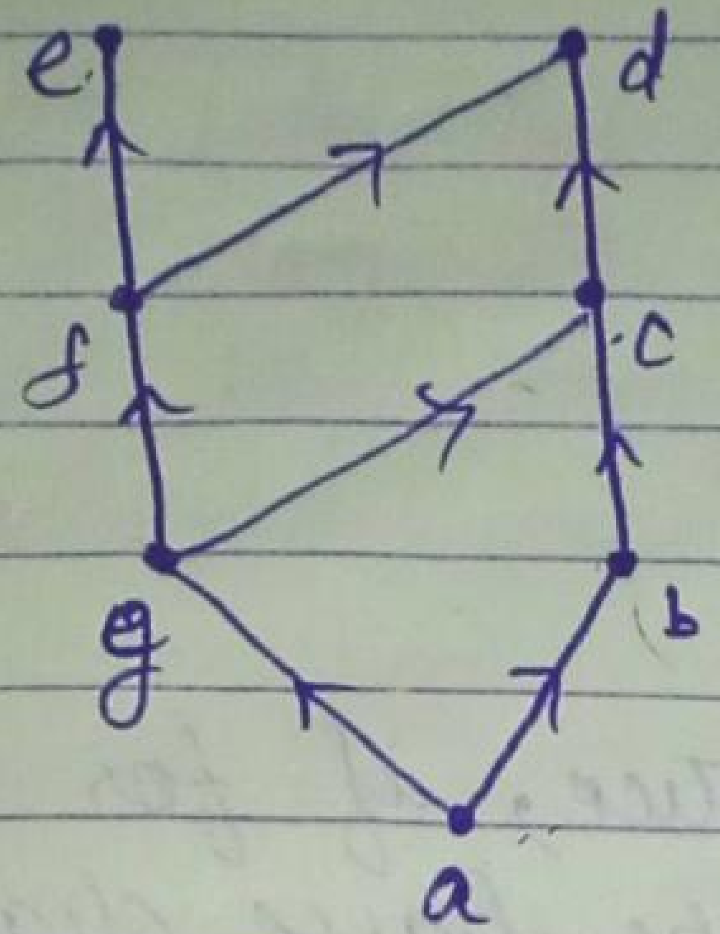
* connected to (a, b, c) and must be the upward elements.

upper bound of (a, b, c) → {d, f, h, j}

→ To first find the upper and lower bound, give direction

25th

Q



upper bounds

Q what are the upper bounds of {a,b}

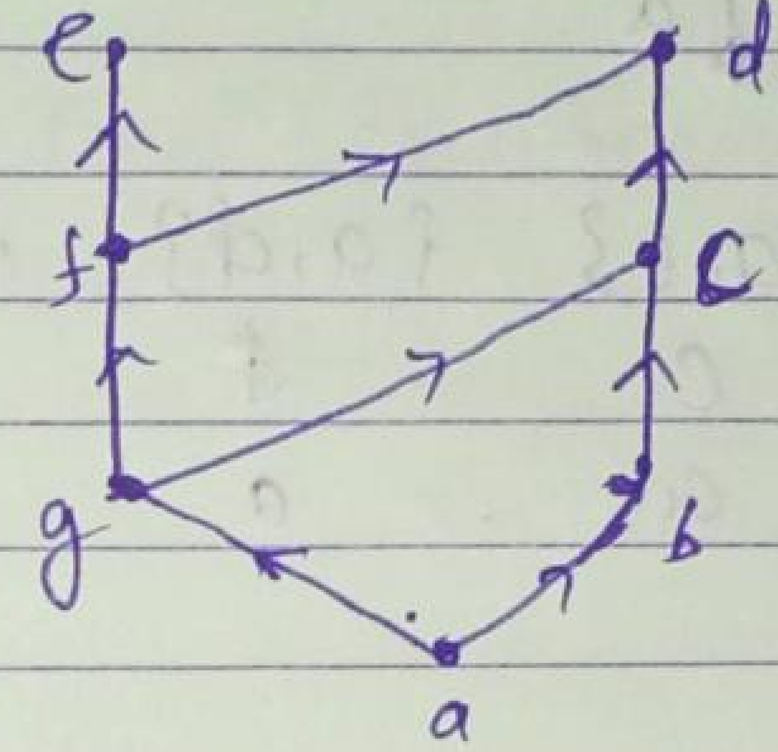
a → {a, b, g, c, d, f, e}

b → {b, c, d}

Find common elements

upper bound {b, c, d}.

Least upper bound (LUB) / Supremum



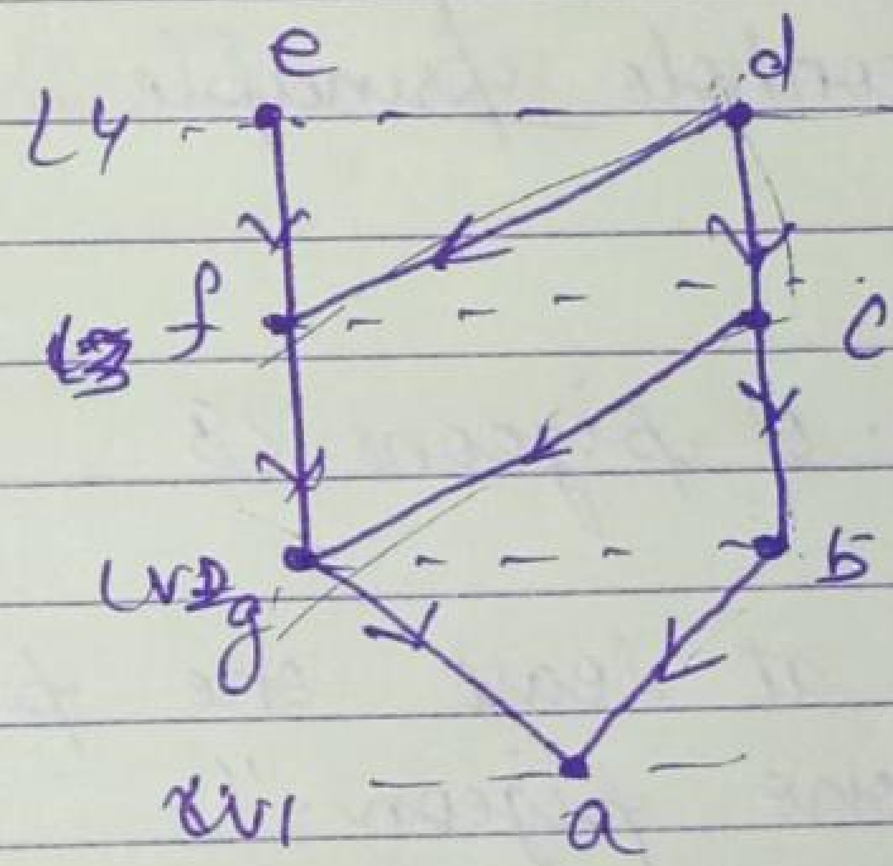
b {b, c, d}

c - {c, d}

LUP - {c, d} - c

~~Lower bound (LUB) / Supromum~~ upper

Least lower {e, d} ? - a



e - {e, g, f, a}

d - {d, c, b, a, f, g}

Lower bound of {f, g, a} = f

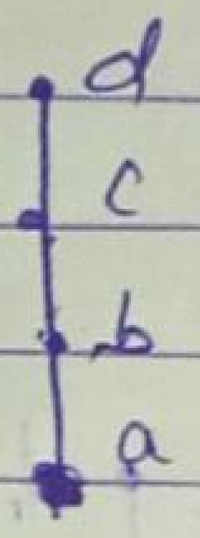
bcoz f is at highest level.

Greatest lower bound or Infemum.

Lattice

Def:- A hasse diagram is called lattice, if for each and every pair of elements of the hasse diagram, LUB and GLB exists.

Ex:-

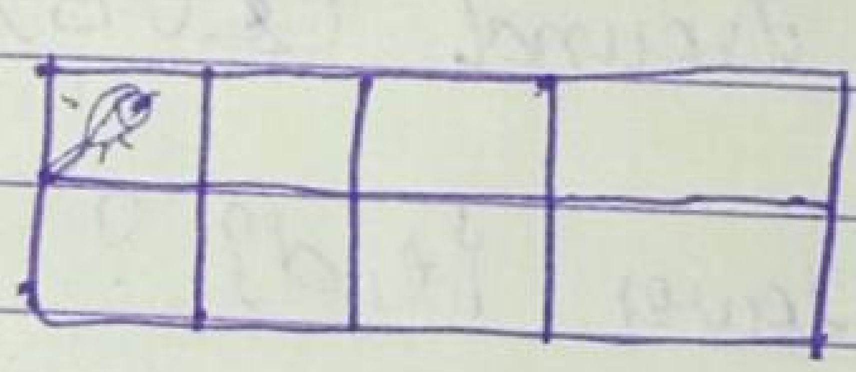


(2 element

Total combination - ${}^4C_2 = 6$

	{a, b}	{a, c}	{a, d}	{b, c}	{b, d}	{c, d}
LUB :	b	c				
GLB :	a	a				

Pigeonhole principle



If No. of pigeons > 8

"then at least one pigeonhole will contain more than ~~two~~ one pigeon."

Def:- If k is a +ve integer and $k+1$ or more object are placed into k boxes, then there is atleast one box containing two or more of the objects.

$\frac{100}{12} = 8$

Ex :- Take 366 days.

If there are 367 people.

Atleast how many people are there who has same b'day ?

→ Ans - (2)

The Generalized Pigeonhole principle

If N objects are placed ~~inside~~ into K boxes, then there is at least one box containing at least $\lceil N/K \rceil$ objects.

Ex:- In a group of 100 people, at least how many of them ~~are~~ who has b'day in same month.

Soln:- Here $n = 100$

$K = 12$

Acc to pigeonhole principle,

Atleast $\frac{100}{12}$ people ~~has~~ b'day fall in same month

$= 8.33$

9

Subtraction Rule / Principle of Inclusion Exclusion

The subtraction rule: If a task can be done in either n_1 ways or n_2 ways then the number of ways to do the task is n_1+n_2 minus the number of ways to do the task that are common to the two different ways.

$A_1 = \{a_1, a_2, a_3, a_4\}$ $\quad |A_1| = 4$

$A_2 = \{a_2, a_5, a_3, a_6, a_7\}$ $\quad |A_2| = 5$

$A_1 \cap A_2 = \{a_2, a_3\}$

~~$|A_1 \cap A_2$~~

$|A_1 \cup A_2| = 4+5-2$ bcoz 2 element are common in both

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$